Title	Operator Theoretical Approach for Transport Equations (SPECTRAL AND SCATTERING THEORY AND RELATED TOPICS)
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Citation	数理解析研究所講究録 (1975), 242: 124-138
Issue Date	1975-06
URL	http://hdl.handle.net/2433/105582
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
242巻 1975年 124-138

Operator theoretical approach for transport equations

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§1. Introduction

The problem of neutron transport in an infinite slab leads, after an appropriate simplification, to the evolution equation

(1) $$\frac{\partial}{\partial t}u(t,x,\mu) = -\mu\frac{\partial}{\partial x}u + \frac{\kappa}{2}\int_{-1}^{1}u(t,x,\mu')d\mu' , \quad t > 0 ,$$

where $u(t,x,\mu)$ is the density of neutrons at x (going in the direction μ at time t) , and κ is a positive parameter. If the slab is extended between the planes $x = -a$, $x = a$ and the outside of the slab is a vacuum, we have the boundary conditions

(2) $$u(t,\mp a,\mu) = 0 , \quad \mu \gtrless 0 , \quad t > 0 .$$

Of course we have to add the initial condition

(3) $$u(0,x,\mu) = u_0(x,\mu) , \quad -a \leq x \leq a , \quad -1 \leq \mu \leq 1 .$$

This equation was deeply studied by J. Lehner and G. M. Wing ([2] - [4]) . In this lecture, a slight improvement will be done.

First we set the problem in an operator-theoretical framework. Put $\mathcal{H} = L^2(-a,a)$, $\mathcal{H} = L^2(-\infty,\infty)$, $M = (-1,1)$, $H = L^2(M\,;\mathcal{H})$ and $H_0 = L^2(M\,;\mathcal{H}_0)$. Define closed linear operators L in $\mathcal{H}$ and A in H (similarly L_0 in $\mathcal{H}_0$ and A_0 in H_0 with $(-a,a)$ replaced by $(-\infty,\infty)$) as follows:

$$D(L) = \{v(x) \in \mathcal{H}\ ;\ \frac{d}{dx}v(x) \in \mathcal{H}\ ,\quad v(-a) = 0\}\ ,$$

$$(Lv)(x) = -\frac{d}{dx}v(x)$$

$$D(A) = \{u(x,\mu) \in H\ ;\ u(\cdot,\mu) \in D(L)\ \text{ for a.e. } \mu > 0\ ,$$
$$u(\cdot,\mu) \in D(L^*)\ \text{ for a.e. } \mu < 0\ ,\quad Au \in H\}\ ,$$

$$(Au)(\cdot,\mu) = \begin{cases} \mu L u(\cdot,\mu)\ , & \mu > 0\ , \\ -\mu L^* u(\cdot,\mu)\ , & \mu < 0\ . \end{cases}$$

Denote by J (resp. $\tilde{J}$) the projection from $\mathcal{H}_0$ to $\mathcal{H}$ (resp. from H_0 to H) , and by K the "integral operator":

$$H \ni u(x,\mu) \longmapsto \frac{1}{\sqrt{2}} \int_{-1}^{1} u(x,\mu)\,d\mu \in \mathcal{H}\ .$$

If we put

(4) $\quad B = A + \kappa K^* K\ ,\quad D(B) = D(A)\ ,$

(5) $\quad B_0 = A_0 + \kappa \tilde{J}^* K^* K \tilde{J}\ ,\quad D(B_0) = D(A_0)\ ,$

then the problem (1)-(3) can be written in an evolution equiation in H :

$$\frac{d}{dt}u = Bu , \quad u(0) = u_0 .$$

Simultaneously we consider the corresponding evolution equation in H_0 :

$$\frac{d}{dt}v = B_0 v , \quad v(0) = v_0 .$$

It is easy to see that L (and hance L^*) generates a contraction semi-group e^{tL} (resp. e^{tL^*}) in $\mathcal{H}$, and L_0 generates an unitary group e^{tL_0} in $\mathcal{H}_0$. Hence A generates a contraction group e^{tA} in H , and A_0 generates an unitary group e^{tA_0} in H_0 . In addition, we obtain that

$$\text{(6)} \qquad e^{tL} = Je^{tL_0}J^* , \quad e^{tL^*} = Je^{-tL_0}J^* \quad (t \geq 0) ,$$

$$\text{(7)} \qquad e^{tA} = \tilde{J}e^{tA_0}J^* , \quad e^{tA^*} = \tilde{J}e^{-tA_0}\tilde{J}^* \quad (t \geq 0) .$$

Since $C = K^*K$ (resp. $C_0 \equiv \tilde{J}^*K^*K\tilde{J}$) is a bounded linear operator in H (resp. H_0) , B (resp. B_0) generates a semi-group e^{tB} in H (resp. a group e^{tB_0} in H_0) . Furthermore we have

$$\text{(8)} \qquad e^{tB} = \tilde{J}e^{tB_0}\tilde{J}^* , \quad t \geq 0 .$$

Following Lehner and Wing, we are concerned with spectral properties of B and B_0, and asymptotic properties of e^{tB} and e^{tB_0}. However the relation (8) implies that there are no essential differences between e^{tB} and e^{tB_0} in the physical meaning. Thus we treat only B_0 and e^{tB_0} in this lecture.

Our main result is as follows:

The continuous spectrum of B_0, which is the whole imaginary axis, is similar to the spectrum of A_0 except for the discrete values of κ.

§2. The spectrum of B_0

Put $\tilde{K} = K\tilde{J}$. Then the second resolvent equation for A_0 and B_0 :

$$(9) \quad (\lambda-B_0)^{-1} = (\lambda-A_0)^{-1} + \kappa(\lambda-A_0)^{-1}\tilde{K}^*\tilde{K}(\lambda-B_0)^{-1}$$

gives the following

$$(10) \quad (\lambda-B_0)^{-1} = (\lambda-A_0)^{-1} + \kappa(\lambda-A_0)^{-1}\tilde{K}^*(1-\kappa G(\lambda))^{-1}\tilde{K}(\lambda-A_0)^{-1} ,$$

where

$$G(\lambda) = \tilde{K}(\lambda-A_0)^{-1}\tilde{K}^* = K\tilde{J}(\lambda-A_0)^{-1}\tilde{J}^*K^* .$$

Thus the study of $G(\lambda)$ is essential for our purpose. Denoting by $\mathbb{B}(\mathcal{H})$ (resp. $\mathbb{C}_\infty(\mathcal{H})$) the set of all bounded (resp. compact) linear operators in $\mathcal{H}$, and by $\|T\|$ the operator norm of $T \in \mathbb{B}(\mathcal{H})$, we summarize some properties of $G(\lambda)$.

Lemma 2.1. (i) $G(\lambda)$ is a $\mathbb{C}_\infty(\mathcal{H})$-valued analytic function in $\mathbb{C}_\pm = \{\lambda ; \operatorname{Re}\lambda \gtrless 0\}$ and satisfies

$$G(\bar{\lambda}) = G(\lambda)^* , \quad G(-\bar{\lambda}) = -G(\lambda)^* .$$

(ii) Let $\lambda \in \mathbb{C}_\pm$. λ belongs to the resolvent set $\rho(B_0)$ of B_0 (i.e., there exists $(\lambda-B_0)^{-1} \in \mathbb{B}(H_0)$) if and only if there exists $(1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$.

(iii) For $\lambda \in \mathbb{C}_+$, $G(\lambda)$ satisfies

$$0 < \mathrm{Re}G(\lambda) = \frac{1}{2}\{G(\lambda) + G(\lambda)^*\} \leq \frac{1}{\mathrm{Re}\lambda} ,$$

$$\mathrm{Im}G(\lambda) = \frac{1}{2i}\{G(\lambda) - G(\lambda)^*\} \lessgtr 0 \qquad (\mathrm{Im}\lambda \gtrless 0) .$$

(iv) For $0 < \beta < \beta'$, $G(\beta) > G(\beta') > G(+\infty) = 0$.

(v) $G(\lambda)$ is continuous in $\mathbb{C}_+ - \{0\} = \{\lambda ; \mathrm{Re}\lambda \geq 0 , \lambda \neq 0\}$ with respect to the norm of $\mathbb{B}(\mathcal{H})$ and satisfies

$$0 < \mathrm{Re}G(\beta+i\gamma) \leq \frac{1}{|\gamma|}(1+\pi) ,$$

$$\mathrm{Im}G(\beta+i\gamma) \gtrless 0 \quad \text{for} \quad \gamma \gtrless 0 \quad \text{and} \quad \beta \geq 0 .$$

(vi) For $\lambda \in \mathbb{C}_+-[0,\infty)$, there exists $(1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$. For any $\delta > 0$, there exists a constant $c_{\kappa,\delta} > 0$ such that

$$\|(1-\kappa G(\lambda))^{-1}\| \leq C_{\kappa,\delta} \qquad (\mathrm{Re}\lambda \geq 0 , \quad |\mathrm{Im}\lambda| \geq \delta) .$$

For $\lambda \in \mathbb{C}_--\{0\}$, there holds

$$\|(1-\kappa G(\lambda))^{-1}\| \leq 1 .$$

For $\beta > 0$, there exists $(1-\kappa G(\beta))^{-1} \in \mathbb{B}(\mathcal{H})$ except for the fimite set of β which depends on κ .

Carrying out simple calculations we obtain

$$G(\lambda) = \int_0^\infty \frac{1}{2}(e^{tL} + e^{tL^*})dt \int_0^1 \frac{1}{\mu} e^{-\frac{\lambda t}{\mu}} d\mu .$$

Using the equality

$$\int_0^1 \frac{1}{\mu} e^{-\frac{z}{\mu}} d\mu = \int_1^\infty \frac{1}{\mu} e^{-\mu z} d\mu$$

$$= -\log z - b + E_0(z) ,$$

where b is Euler number and $E_0(z)$ is an entire analytic function of z which satisfies $|E_0(z)| \leq |z|$ for $z \in \mathbb{C}_+$, we have

(11) $G(\lambda) = \int_0^\infty \mathrm{Re}\, e^{tL}\{-\log \lambda t - b - E_0(\lambda t)\}dt .$

We put

$$K(\lambda) = -\int_0^\infty \mathrm{Re}\, e^{tL} dt(\log\lambda + b) + \int_0^\infty \mathrm{Re}\, e^{tL}(-\log t)dt ,$$

$$G_0(\lambda) = \int_0^\infty \mathrm{Re}\, e^{tL} E_0(\lambda t)dt .$$

Since $\int_0^\infty \mathrm{Re}\, e^{tL} dt = \mathrm{Re}\, L^{-1}$ reduces to the 1-dimensional operator:

$$\mathcal{H} \in u(x) \longmapsto \frac{1}{2} \int_{-a}^{a} u(x)dx = a\frac{1}{2a}(u,1)1 \in \mathcal{H} ,$$

we have

(12) $K(\lambda) = -aN\log \lambda - baN + K_0$

where N is the orthogonal projection $\frac{1}{2a}(\ ,1)1$ in $\mathcal{H}$ and

$$K_0 = \int_0^\infty \mathrm{Re}\, e^{tL}(-\log t)dt \in \mathbf{C}_\infty(\mathcal{H}) .$$

The inequality $|E_0(z)| \leq |z|$ $(z \in \overline{\mathbb{C}}_+)$ implies that

$$\|G_0(\lambda)\| \leq \int_0^a |\lambda t| dt = \frac{a^2}{2}|\lambda| .$$

This implies that the spectrum $\sigma(G(\beta))$ of $G(\beta)$ converges to the spectrum $\sigma(K(\beta))$ of $K(\beta)$ as $\beta \downarrow 0$. Thus we have the following

Lemma 2.2. Let $\{\rho_n(\beta)\}$ be the set of (positive) eigen values of $G(\beta)$ (counted as many times as multiplicities). We can arrange $\{\rho_n(\beta)\}$ in the following way;

$\rho_n(\beta)$ is monotone decreasing in $\beta \in (0,\infty)$,

$\rho_n(\beta) \downarrow 0$ as $\beta \uparrow \infty$,

$\rho_n(\beta) \uparrow \rho_n^*$ as $\beta \downarrow 0$,

$\rho_n(\beta)$ is real analytic in $\beta \in (0,\infty)$.

Here $\rho_1^* = \infty$ and $\rho_2^* \geq \rho_3^* \geq \cdots$ are the eigen values of $N'K_0N'$ arranged in the decreasing order. (In above we have put $N' = 1 - N$. Note that $N'K_0N' > 0$ on the range $R(N')$ of N'.)

For $\kappa > 0$, denote by $N(\kappa)$ the number of ρ_n^* such that $\kappa\rho_n^* > 1$. Let $\beta_n = \beta_n(\kappa)$ be the root of $\kappa\rho_n(\beta) = 1$ for $n = 1, \cdots, N(\kappa)$. Then $(1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$ exists for $\lambda \in \overline{\mathbb{C}}_- \cup \overline{\mathbb{C}}_+ - \{0, \beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$. The $\beta_n(\kappa)$'s are simple roots of $(1-\kappa G(\lambda))^{-1}$. Hence $(\lambda - B_0)^{-1} \in \mathbb{B}(\mathcal{H})$ exists for $\lambda \in \mathbb{C}_- \cup C_+ - \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$ and has simple poles at $\{\beta_1(\kappa), \cdots \beta_{N(\kappa)}(\kappa)\}$. A simple argument connected with Lemma 2.1 shows that the there is not the point spectrum $\sigma_p(B_0)$ of B_0 on the imaginary axis $i\mathbb{R}$. Hence $\sigma_p(B_0)$ coincides with the discrete spectrum $\sigma_d(B_0)$ of B_0, i.e. $\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_n(\kappa)\}$. Similarly $\sigma_p(B_0^*) = \sigma_d(B_0^*) = \{\beta_n(\kappa)\}$. Furthermore the inequality (proved by Ukai)

$$\begin{aligned}\mathrm{Re}(\tilde{K}^*u,\ (\lambda-A_0)^{-1}\tilde{K}^*u) &\geq \mathrm{Re}((\lambda-A_0)(\lambda-A_0)^{-1}\tilde{K}^*u,\ (\lambda-A_0)^{-1}\tilde{K}^*u) \\ &= \mathrm{Re}\lambda\|(\lambda-A_0)^{-1}\tilde{K}^*u\|^2\end{aligned}$$

shows that for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}\|(\lambda-A_0)^{-1}\tilde{K}^*u\|^2 &\leq \frac{1}{\mathrm{Re}\lambda}\mathrm{Re}(u,G(\lambda)u) \\ &\leq \frac{1}{\mathrm{Re}\lambda}\|u\|\|G(\lambda)u\| .\end{aligned}$$

Thus the compactness of $G(\lambda)$ implies that of $(\lambda-A_0)^{-1}\tilde{K}^*$. This implies that the essential spectrum of B_0 coincides with that of A_0, which is the whole imaginary axis. All these arguments show that the continuous spectrum $\sigma_0(B_0)$ of B_0 is the imaginary axis $i\mathbb{R}$, and the residual spectrum $\sigma_r(B_0)$ of B_0 is empty. Thus we have the following theorem due to Lehner.

Theorem 1. Let $\kappa > 0$ and B_0 be defined by (5). Then

$$\rho(B_0) = \mathbb{C}_- \cup \mathbb{C}_+ - \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$$

$$\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$$

$\sigma_c(B_0) = i\mathbb{R}$, $\sigma_r(B_0) = \phi$

$(\lambda - B_0)^{-1}$ has simple poles at $\{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\}$.

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§3. The similarity of the continuous spectra of A_0 and B_0

Denote by $P_j = P_j(\kappa)$ the residue of $(\lambda-B_0)^{-1}$ at $\lambda = \beta_j(\kappa)$, that is the eigen projection of B_0 belonging to $\beta_j(\kappa)$, $j = 1,\cdots,N(\kappa)$. Put $Q_1 = \Sigma P_j$, $Q_2 = 1 - Q_1$, $B_1 = B_0 Q_1$ and $B_2 = B_0 Q_2$. Then $(\lambda-B_0)^{-1}Q_2 = (\lambda-B_2)^{-1}Q_2$ is analytic in $\mathbb{C}_\pm$ and there hold

$$(\lambda-B_0)^{-1} = (\lambda-B_0)^{-1}Q_2 + \sum_{j=1}^{N(\kappa)} \frac{1}{\lambda-\beta_j} P_j ,$$

$$e^{tB_0} = e^{tB_0}Q_2 + \Sigma e^{t\beta_j}P_j .$$

In order to study the spectral property of B_2, we use the method of A_0-smooth perturbation developed by Kato [1]. In what follows, we put for a fixed $\alpha \in (0, 1)$

$$\alpha_1(s) = \begin{cases} 2^\alpha - \log|s| , & |s| \leq 1 , \\ (1+|s|)^\alpha , & |s| \geq 1 , \end{cases}$$

$$\alpha_2(s) = (1+|s|)^\alpha ,$$

and for later conveniens $N_1 = N$ and $N_2 = N'$. From Lemma 2.1, (11) and (12), we obtain for some constant a_0

$$\| \mathrm{Re}\, N_j G(\pm\sigma+i\gamma)N_j \| \leq \frac{1}{2} a_0 \alpha_j(\gamma)^{-1} , \quad j = 1,2 .$$

Let $\{E_0(s)\}$ be the spectral resolution of $-iA_0$ and put $R(\lambda)$

$= (\lambda - A_0)^{-1} = \int (\lambda - is)^{-1} dE_0(s)$. Following Kato [1] , we have

$$\| N_j\tilde{K}(\lambda - A_0)^{-1}u - N_j\tilde{K}(-\bar{\lambda} - A_0)^{-1}u\|^2$$

$$\leq 2\|\mathrm{Re}\ N_jG(\lambda)N_j\| (\{(\lambda - A_0)^{-1} - (-\bar{\lambda} - A_0)^{-1}\}u,u)$$

$$\leq a_0\alpha_j(\gamma)^{-1} \int_{-\infty}^{\infty} \frac{2\sigma}{\sigma^2 + (\gamma - s)^2} d\|E_0(s)\|^2 , \quad \lambda = \sigma + i\gamma .$$

This implies

$$\int_{-\infty}^{\infty} \alpha_j(\gamma) \| N_j\tilde{K}R(\sigma + i\gamma)u - N_j\tilde{K}R(-\sigma + i\gamma)u\|^2 d\gamma$$

$$\leq 2\pi a_0\|u\|^2 , \quad j = 1,2 .$$

Using estimates for Hilbert transforms with weighted norms, we have

$$\int_{-\infty}^{\infty} \alpha_j(\gamma)\| N_j\tilde{K}R(\sigma + i\gamma)u\|^2 d\gamma$$

$$\leq C_0 \int_{-\infty}^{\infty} \alpha_j(\gamma)\| N_j\tilde{K}R(\sigma + i\gamma)u - N_j\tilde{K}R(-\sigma + i\gamma)u\|^2 d\gamma$$

$$\leq 2\pi a_0C_0\|u\|^2 ,$$

Hence $N_j\tilde{K}R(\sigma + i\gamma)u$ is an element of a $\mathcal{H}$-valued Hardy class with a weighted norm, and is a continuous function of $\sigma \geq 0$ and $\sigma \leq 0$ with values in $L^2(\mathbb{R}, \alpha_j(\gamma)^{\frac{1}{2}} d\gamma ; \mathcal{H})$.
Putting $R_1(\lambda) = (\lambda - B_0)^{-1}$ and recalling that

$$\tilde{K}(\lambda - B_0)^{-1} = (1 - \kappa G(\lambda))^{-1}\tilde{K}(\lambda - A_0)^{-1} ,$$

we define so called wave operators $W_\pm$ and $Z_\pm$ as follows:

$$(W_{\pm u,v}) = (u,v) \pm \frac{\kappa}{2\pi i}\int_{-\infty}^{\infty}(\widetilde{K}R(\pm 0+i\gamma)u,\ \widetilde{K}R_1(\mp i0+i\gamma)^*\ v)d\gamma\ ,$$

$$(Z_{\pm u,v}) = (Q_2u,v) \mp \frac{\kappa}{2\pi i}\int_{-\infty}^{\infty}(\widetilde{K}R_1(\pm 0+i\gamma)Q_2u,\ \widetilde{K}R(\mp 0+i\gamma)^*v)d\gamma\ .$$

To see the convergence of these integrals, we have to investigate the behavior of $(1-\kappa G(\lambda))^{-1}$ near $\lambda = \pm 0 \in \mathbb{C}_\pm$. We put $N_iG_{ij}(\lambda)N_j = G_{ij}(\lambda)$, $i = 1,2$. Then $G_{ij}(\lambda)$'s have the following forms:

$$G_{11}(\lambda) = \{-a\log\lambda - ab - g_1(\lambda)\}N_1\ ,$$

$$G_{12}(\lambda) = G_{21}(\bar{\lambda})^* = N_1K_0N_2 + N_1G_0(\lambda)N_2\ ,$$

$$G_{22}(\lambda) = N_2K_0N_2 + N_2G_0(\lambda)N_2\ ,$$

$$|g_1(\lambda)| \le \frac{1}{2}a^2|\lambda|\ , \quad \|N_iG_0(\lambda)N_j\| \le \frac{1}{2}a^2|\lambda|\ .$$

Let us assume that $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2K_0N_2)$. Then for sufficiently small $\lambda \in \mathbb{C}_+$, there exists $(1 - \kappa G_{22}(\lambda))^{-1} \in \mathbb{B}(\mathcal{H})$ with uniformly bounded norm. Hence we have

$$\|(1 - \kappa G(\lambda))^{-1}u\| \le \frac{c_1}{2-\log|\lambda|}\|N_1u\| + c_2\|N_2u\|$$

for sufficiently small $\lambda \in \mathbb{C}_+$ (and hence for small $\lambda \in \mathbb{C}_-$) . This implies

$$\|\widetilde{K}R_1(\lambda)u\| \le \frac{c_1}{2-\log|\lambda|}\|N_1\widetilde{K}R(\lambda)u\| + c_2\|N_2\widetilde{K}R(\lambda)u\|$$

for sufficiently small $\lambda \in \mathbb{C}_\pm$. Thus the above integrals converge absolutely, and $W_\pm$, $Z_\pm \in B(H_0)$. Following Kato's argument, we can easily see that

(13) $Z_\pm W_\pm = 1$, $W_\pm Z_\pm = Q_2$

$$(\lambda - B_2)W_\pm = W_\pm(\lambda - A_0)^{-1} \quad \text{i.e.} \quad B_2 = W_\pm A_0 Z_\pm .$$

(14) $e^{tB_2} = W_\pm e^{tA_0} Z_\pm$.

Thus we have

Theorem 2. Let $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2K_0N_2)$. Then A_0 and $B_2 = B_0Q_2$ are similar to each other. That is, $W_\pm$ and $Z_\pm \in \mathbb{B}(H_0)$ exist and satisfy (13) and (14). Furthermore we have

$$W_\pm = s - \lim_{t\to\pm\infty} Q_2 e^{tB_0} e^{-tA_0} ,$$

$$Z_\pm = s - \lim_{t\to\pm\infty} e^{tA_0} e^{-tB_0} Q_2 .$$

If we put $F(\Delta) = W_\pm(\Delta)E_0(\Delta)Z_\pm(\Delta)$, $\Delta = (a,b)$, then $F(\Delta)$ is the "spectral resolution" of B_2 , i.e.,

$$B_0 = i \int_{-\infty}^{\infty} \lambda dF(\lambda) + \Sigma\beta_j P_j .$$

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